Interacting diffusions in a random medium: comparison and longtime behavior

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Abstract

We consider a collection of linearly interacting diffusions (indexed by a countable space) in a random medium. The diffusion coefficients are the product of a space-time dependent random field (the random medium) and a function depending on the local state. The main focus of the present work is to establish a comparison technique for systems in the same medium but with different state dependence in the diffusion terms. The technique is applied to generalize statements on the longtime behavior, previously known only for special choices of the diffusion function.

One of these special choices, which we employ as a reference model, is that of interacting Fisher-Wright diffusions in a catalytic medium where duality was used to obtain detailed results. The other choice is that of interacting Feller's branching diffusions in a catalytic medium which is itself an (autonomous) branching process and where infinite divisibility was used as the main tool.

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1 Introduction and Main Results

1.1 Background

In this paper we are concerned with the construction and the longtime behavior of systems of countably many interacting diffusions, where the diffusion function of the state of one component depends only on that state and on an autonomously fluctuating medium and may therefore be *varying* both in *space and time*. The interaction between components is linear and time- and space-homogeneous. Its precise form is motivated either by the population dynamics notion of migration or by the population genetics notion of "choice of ancestors from other colonies". The systems have features which are different from the classical time-homogeneous case.

We start with constructing the models and establishing a comparison principle. This comparison principle is useful because it transfers results previously known only in special cases, in particular for population models, to a wider class of interacting diffusions. The case of branching systems has already been studied in special contexts: super Brownian motion in a catalytic medium is discussed in a sequence of papers by Dawson and Fleischmann [DF94, DF95, DF97a, DF97b] and Fleischmann and Klenke [FK99, FK00], while related particle models are studied by Greven, Klenke and Wakolbinger [GKW99]. In [GKW01] another principal population model, the case of interacting Fisher-Wright diffusions, is studied as a prototype for so-called resampling models.

The longtime behavior of interacting systems reflects the competition between *migration* and the *fluctuations* in the components. Depending on the parameters either the migration dominates, resulting in limiting states which are spatially constant, or the diffusion dominates, resulting in degenerate states concentrated on the traps of the pure diffusion, or both mechanisms are relevant in the longrun. In the latter case we get a non-degenerate limiting behavior with either limiting laws concentrated on states, which are constant but not concentrated only on the traps, or we get an equilibrium state with a (nontrivial) local dependence structure.

The new phenomenon due to the random medium is that the regime with nontrivial limiting behavior splits into two cases, one has an equilibrium which is spatially exchangeable, the other one has a local dependence structure. In the case of media which are given by a voter model or by Feller's branching diffusions we can characterize the exchangeable states of the process in the random medium quite explicitly due to the knowledge one has about the cluster formation of these systems in the homogeneous case, see [GKW99], [GKW01], [FK99], [DF94, DF95, DF97a, DF97b]. This raises the question to what extent these results are valid in larger classes of models.

We will establish in this paper that the just described pattern of behavior is fairly independent of the special nature of the fluctuations and occurs therefore in a larger class of systems. A major tool hereby is a comparison result of Cox, Fleischmann and Greven [CFG96] for systems of interacting diffusions, which we will extend to the case of time-inhomogeneous and site-dependent diffusion coefficients.

1.2 Construction of the models and special cases of particular interest

We introduce a process with countably many components driven by a space-time inhomogeneous diffusion mechanism and interacting via a linear coupling. Consider the system $X = (X_t)_{t\geq 0}$ with

$$X_t = (X_t(i))_{i\in S} \in \mathbb{E} \subseteq I^S, \tag{1.1}$$

where $I \subset [0,\infty)$ is either a closed interval or $[0,\infty)$ itself and S is a countable or finite set, $\mathbb{E}$ is defined in (1.4) below and X is defined by the following system of stochastic differential equations (SSDE):

$$dX_t(i) = \mathcal{A}X_t(i)dt + \sqrt{2g_{i,t}(X_t(i))}dW_t(i), \quad i \in S\,, \quad X_0 = x_0 \in \mathbb{E}. \tag{1.2}$$

The ingredients of this equation are the following:

(i) $\mathcal{A}(i,j)$ or $\mathcal{A}^T(i,j) = \mathcal{A}(j,i)$ is the q–matrix of a rate 1 Markov chain on S. The former case arises in spatial population genetics models where $X_t(i)$ stands for the proportion of a certain type at site i and where, in addition to local resampling, ancestors are chosen from other sites according to $\mathcal{A}$. The latter case refers to migration of particles according to $\mathcal{A}$ as considered in population dynamics models where $X_t(i)$ measures the number of particles at site i. Note that only in this case the total mass $\langle X_t, 1\rangle$ is a martingale (if finite).

We write $a_t = \exp(t\mathcal{A})$, $t \geq 0$ for the semigroup generated by $\mathcal{A}$.

(ii) $\{(W_t(i))_{t\geq 0}, i \in S\}$ is an independent family of standard Brownian motions.

(iii) The collection $\{(g_{i,t})_{t\geq 0}, i \in S\}$ of diffusion functions each taking values in $[0,\infty)$ satisfies

$$
\begin{aligned}
&\bullet\ g_{i,t} \text{ is locally Lipschitz continuous for all } i \in S,\ t \geq 0,\\
&\bullet\ t \mapsto g_{i,t}(x) \text{ is in } D([0,\infty),\mathbb{R}) \text{ and is continuous except at isolated points, for all}\\
&\quad\ i \in S, \quad x \in \mathbb{R},\\
&\bullet\ g_{i,t}(x) = 0 \text{ for } x \text{ in the boundary of } I,\\
&\bullet\ \text{for all } T > 0 \text{ and } i \in S, \text{ there exists a constant } C_T(i) < \infty \text{ such that}\\
&\quad\ g_{i,T}(x) \leq C_T(i)(1 + x^2).
\end{aligned}
\tag{1.3}
$$

(iv) The state space $\mathbb{E}$ is defined as a Liggett-Spitzer space (see [LS81]):

$$\mathbb{E} = L_1(S,\gamma) = \left\{x \in I^S : \langle \gamma, x\rangle < \infty\right\}, \tag{1.4}$$

where γ is a *strictly positive* measure on S satisfying

$$\gamma\mathcal{A} \leq M\gamma, \tag{1.5}$$

for some $M < \infty$. (Note that such a γ always exists (cf. [LS81] or [CFG96]). Also note that if I is a bounded interval then (1.4) is void since we can pick for γ any finite strictly positive measure.)

We are now ready to show that our process X is well-defined.

Theorem 1 (Existence and Uniqueness) *There exists a unique strong solution X of (1.2) with $X_t \in \mathbb{E} \quad \forall\, t \geq 0$. This process X is a Markov process.*

Remark 1.1 If we allow in (1.2) initial conditions $X_s = x_0 \in \mathbb{E}$ for all s then the process X is a strong Markov process.

In the rest of this paper we focus on a special case of particular interest, which is of the following form. We think of X as a process in a *randomly fluctuating medium*, where the medium defines for example *branching* or *resampling rates* which are varying in time and space. However the randomness of theses rates we can bring in later. The basic set-up in this case is therefore the following. We are given a collection

$$\{(H_t(i))_{t\geq 0}, \quad i \in S\} \tag{1.6}$$

of functions (the rates) which are piecewise continuous. Then we put:

$$g_{i,t}(x) = H_t(i)g(x), \qquad i \in S,\ t \geq 0, \tag{1.7}$$

where g is a function satisfying

$$
\begin{aligned}
&\bullet\ g \text{ is locally Lipschitz},\\
&\bullet\ g(x) = 0 \text{ for } x \text{ in the boundary of } I,\\
&\bullet\ g(x) \leq C(1 + x^2) \qquad \forall\, x \in I.
\end{aligned}
\tag{1.8}
$$

This construction should be viewed as simply choosing in every point a time and space dependent constant in front of g. This constant will be generated by a random process H which evolves autonomously. To be clear at this important point, the construction of our model can be viewed as a two–stage experiment:

(1) Choose a realization of H.

(2) For given H sample X.

(Note that for every $T > 0$, $i, j \in S$, and fixed $(H_k(t),\, t \le T)$, $k \neq i$, one can show that the random variable $(X_j(t),\, t \le T)$ is a continuous function of $(H_i(t),\, t \le T)$, considered as a map from $L_1([0,T]) \to L_1([0,T]\times\Omega)$, where Ω is the underlying probability space. In particular $\mathcal{L}[X|H]$ is a measurable function of H.)

Two particular choices for g and H are of special interest since they are easier to study and thus will serve later as the reference models for the comparison arguments.

Examples (Fisher-Wright and Branching diffusions with time-space fluctuating rates)

Of particular interest for both applications and as mathematical tool are the following two choices of g for the process X itself, which specializes (1.7):

$$I = [0,1], \quad g^c_{FW}(x) = c \cdot x(1-x), \tag{1.9}$$

$$I = [0,\infty), \quad g^c_B(x) = c \cdot x, \tag{1.10}$$

($c > 0$ a constant). The functions g^c_{FW} and g^c_B are known as the diffusion coefficients for the Fisher–Wright diffusion and Feller's branching diffusion, respectively.

Next we need to specify the medium in these cases. A typical situation in the context of (1.9) or (1.10) uses for the process generating the medium (that is, for the diffusion function) the following: the process H is itself a solution of our SSDE of the type (1.2) with $g_{i,t}(x) = \widetilde{g}(x)$ for all $i \in S$ and $t \ge 0$ where $\widetilde{g}$ satisfies the requirements of (1.8).

If $\widetilde{g}(x) = x$, that is, in the situation (1.10), one obtains a *reactant-catalyst system*, the branching diffusion $H_t = (H_t(i),\ i \in S)$ describes the mass of the catalyst at i at time t and the process X the mass of the reactant. In the context of (1.9) one has at fixed population size two types of reactants and the process X describes the relative proportion of one of them.

Another important choice is $\tilde{g}(x) = x(1-x)$. In this case the catalyzing system involves two types of which only one is able to catalyze. For technical reasons (cf. [GKW01]) we will consider here the case where H is given by a *voter model* on S, which can be viewed as the limiting dynamics of g^c_{FW} as $c \to \infty$.

1.3 A comparison theorem for time-inhomogeneous interacting diffusions

We continue with preparing a tool for the analysis of the above introduced models. The goal is to compare the distributions of two processes X^1 and X^2 which satisfy the following three conditions:

(i) they start in the same initial point $X_0 \in \mathbb{E}$,

(ii) they both evolve according to (1.2), but based on two *different* collections of diffusion functions $\{(g^1_{i,t}(x))_{t\ge 0},\ i \in S, x \in I\}$ and $\{(g^2_{i,t}(x))_{t\ge 0},\ i \in S, x \in I\}$ respectively,

(iii) those collections of diffusion functions satisfy

$$g^1_{i,t} \ge g^2_{i,t} \qquad \forall\, i \in S,\ t \ge 0. \tag{1.11}$$

The idea is now that (compare [CFG96]) *more noise* in the system, in the sense of (1.11), means a *more spread out distribution* of the process X^1 compared to X^2 at every time point. To define properly the notion of one distribution to be "more spread out" than another one, we use cones of functions. The set of all nonnegative convex functions on I^S looks like a natural candidate. However this class is for our purposes too small since it does not have the needed conservation properties under the semigroup of the evolution.

Let $C_{2,b,f}(\mathbb{E})$ denote the space of bounded twice continuously differentiable functions $F:\mathbb{E}\to\mathbb{R}$ with bounded first and second derivatives such that F depends on only finitely many coordinates. Further denote by $C^+_{2,b,f}(\mathbb{E})$ the subspace of nonnegative functions, and write D_i for the partial derivative with respect to the component at site i.

Definition 1.2 (Function cone) *We introduce the following cones of functions:*

$$\begin{aligned} \mathcal{F} &:= \{F\in C^+_{2,b,f}(\mathbb{E}):\ D_iD_jF\geq 0,\ i,j\in S\},\\ \mathcal{F}^{\uparrow} &:= \{F\in\mathcal{F}:\ D_iF\geq 0,\, i\in S\},\\ \mathcal{F}^{\downarrow} &:= \{F\in\mathcal{F}:\ D_iF\leq 0,\, i\in S\}. \end{aligned} \tag{1.12}$$

Note that $\mathcal{F}=\mathcal{F}^{\downarrow}$ if $I=[0,\infty)$ since $f\in\mathcal{F}$ is bounded and convex in each coordinate.

The cone $\mathcal{F}$ of functions is most suited for systems where the single components take values in $[0,\infty)$ or in a bounded interval. (For systems whose components take values in all of $\mathbb{R}$ requirements of convexity and boundedness like in the definition of $\mathcal{F}$ would be difficult to reconcile – that is why we made the global assumption $I\subset[0,\infty)$.)

Important examples for functions in $\mathcal{F}$ are:

(i) For $\lambda\in[0,\infty)^S$, and λ vanishing outside a finite set, define $F_\lambda\in\mathcal{F}^{\downarrow}$ by

$$F_\lambda(\xi)=\exp(-\langle\lambda,\xi\rangle). \tag{1.13}$$

(ii) If I is bounded, then for $i,j\in S$ a function $F_{i,j}\in\mathcal{F}^{\uparrow}$ can be defined by $F_{i,j}(\xi)=\xi(i)\xi(j)$.

In the set-up just described we prove for systems given via (1.2):

Theorem 2 (Comparison)
Assume X^l, $l=1,2$ are processes as described at the beginning of this subsection.

(a) If $F\in\mathcal{F}$ then

$$\mathbf{E}[F(X^1_t)]\geq\mathbf{E}[F(X^2_t)]\qquad\forall\, t\geq 0. \tag{1.14}$$

(b) If $F_k\in\mathcal{F}^{\uparrow},\ k=1,\dots,n$ or $F_k\in\mathcal{F}^{\downarrow},\ k=1,\dots,n$ then for $0\leq t_1\leq\cdots\leq t_n<\infty$

$$\mathbf{E}[F_1(X^1_{t_1})\cdots F_n(X^1_{t_n})]\geq\mathbf{E}[F_1(X^2_{t_1})\cdots F_n(X^2_{t_n})]. \tag{1.15}$$

Remark 1.3 In the case $I=[0,\infty)$, the theorem is easily generalized to unbounded functions, such as polynomials if X has sufficiently high moments.

Remark 1.4 The proof of (a) will be based on showing that $\mathcal{F}$ is preserved under the dynamics of X. Thus monotonicity yields that also $\mathcal{F}^{\uparrow}$ and $\mathcal{F}^{\downarrow}$ are preserved. In order to show (b), one can proceed as in [CFG96]. We only give a short outline here that makes clear why the F_i have to be in $\mathcal{F}^{\uparrow}$ or $\mathcal{F}^{\downarrow}$.

By an induction argument it is enough to show that $F_1\cdot G_2\in\mathcal{F}$, where $G_2(x):=\mathbf{E}[F_2(X_t)|X_0=x]$. Now

$$D_iD_jF_1G_2=G_2D_iD_jF_1+F_1D_iD_jG_2+(D_iF_1)(D_jG_2)+(D_jF_1)(D_iG_2).$$

The first two terms are non-negative since $F_1,G_2\in\mathcal{F}$. Now use the assumption $F_1,F_2\in\mathcal{F}^{\uparrow}$ or $F_1,F_2\in\mathcal{F}^{\downarrow}$ (which implies $G_2\in\mathcal{F}^{\uparrow}$ or $G_2\in\mathcal{F}^{\downarrow}$ respectively). Hence also the third and fourth term are non-negative.

1.4 Application of the comparison result to the longtime behavior

The most important use of the comparison theorem is to verify universality properties in the longterm behavior. We continue with systems of the form $g_{i,t} = H_t(i)g$. However in this section we change a bit the point of view. Our main interest is now in the law of the bivariate process of the interacting diffusion X together with the medium H.

As our first application we determine the longtime behavior of interacting diffusions in a voter-medium, where the index set S is $\mathbb{Z}^2$ or $\mathbb{Z}$ and the diffusion function g is quite general.

Recall that the voter model is a $\{0,1\}^{\mathbb{Z}^d}$-valued Markov process evolving as follows:

(i) Each site has an independent clock ringing after successive independent exponential waiting times.

(ii) Whenever the clock rings at a site i one of the nearest neighbors of i is chosen at random and i assumes the value of that neighbor.

We will henceforth assume that the q–matrix $\mathcal{A}$ of X is also nearest neighbor.

Assume that $I = [0,1]$ and $g_{i,t}(x) = H_t(i)g(x)$, where the process H is a voter model with

$$\mathcal{L}[H_0] = \pi_{\theta_1}, \tag{1.16}$$

where $\pi_{\theta_1} = ((1-\theta_1)\delta_0 + \theta_1\delta_1)^{\otimes\mathbb{Z}^d}$. We start X in the constant state:

$$X_0 = \theta_2 \mathbb{1}. \tag{1.17}$$

In dimension 1 and 2 the homogeneous system converges in law to $\theta_2\delta_{\mathbf{1}} + (1-\theta_2)\delta_{\mathbf{0}}$. The same holds for the voter model with θ_2 replaced by θ_1. In dimension $d \geq 3$ both systems approach equilibrium states which are translation invariant and shift ergodic with densities θ_2 respectively θ_1. The behavior for $d = 1, 2$ is very different in random medium.

We begin describing the features which are common to both $d = 1$ and $d = 2$.

Theorem 3 (Voter medium, $I = [0,1]$) *Assume that $d = 1$ or $d = 2$. Then there exist (nontrivial) $[0,1]$–valued random variables $\widetilde{H}$ and $\widetilde{X}$ such that for all choices of the diffusion functions g in (1.7) which are Lipschitz functions on $[0,1]$ with $g(0) = g(1) = 0$, $g(x) > 0$ for $x \in (0,1)$*

$$\mathcal{L}_{\pi_{\theta_1}\otimes\delta_{\theta_2\mathbb{1}}}\left[(H_t, X_t)\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}\left[(\widetilde{H}\mathbb{1}, \widetilde{X}\mathbb{1})\right]. \tag{1.18}$$

(By $\Longrightarrow$ we denote weak convergence of probability measures, where we assume the space $\{0,1\}^{\mathbb{Z}^d} \times [0,1]^{\mathbb{Z}^d}$ to be equipped with the product topology. This topology is equivalent to the topology induced by the Liggett-Spitzer norm since the coordinates are bounded. Hence (1.18) is a statement about convergence of finite dimensional distributions.)

Next we identify the limiting laws appearing on the r.h.s. of (1.18) separately in the cases $d = 1$ and $d = 2$. We shall derive both these theorems with the help of Theorem 2 part (a) from results in [GKW01], where the assertion was proved for the case $g = g^c_{FW}$.

Theorem 4 (Voter medium, $I = [0,1]$) *For $d = 2$ the random variables $\widetilde{H}$ and $\widetilde{X}$ can be represented as*

$$\widetilde{H} = Y^1_\infty, \qquad \widetilde{X} = Y^2_{\int_0^\infty p(Y^1_s)ds}, \tag{1.19}$$

where Y^1 and Y^2 are two independent standard Fisher-Wright diffusions starting in $Y^1_0 = \theta_1$ and $Y^2_0 = \theta_2$ and $p : [0,1] \to [0,1]$ is the unique solution of

$$\begin{aligned} p''(x) &= -2\frac{p(x)(1-p(x))}{x(1-x)}, \quad x \in (0,1), \\ p(0) &= 0, \quad p(1) = 1. \end{aligned} \tag{1.20}$$

Consider now the case $d = 1$. The result in (1.18) in $d = 1$ can be explained and the limiting law can be calculated via a stronger statement namely a rescaling result, which we can obtain using now part (b) of our Theorem 2.

Theorem 5 (Voter medium, $I = [0, 1]$) *For $d = 1$*

$$\mathcal{L}_{\pi_{\theta_1}\otimes\delta_{\theta_2\mathbb{1}}}\left[\left(\left(H_{st}(\lfloor z\sqrt{t}\rfloor), X_{st}(\lfloor z\sqrt{t}\rfloor)\right)_{z\in\mathbb{R}}\right)_{s\geq 0}\right] \underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathcal{L}\left[(H^\infty, X^\infty)\right], \tag{1.21}$$

where the limiting process $((H_s^\infty(z), X_s^\infty(z))_{z\in\mathbb{R}})_{s\geq 0}$ is independent of the choice of g.

The case $d \geq 3$ can be treated with coupling techniques for all g so that we get automatically the universality in the behavior for $t \to \infty$. This was explained in [GKW01] and gives here that for all g as in Theorem 3:

$$\mathcal{L}_{\pi_{\theta_1}\otimes\delta_{\theta_2\mathbb{1}}}\left[(H_t, X_t)\right] \overset{t\to\infty}{\Longrightarrow} \nu_\theta, \quad \theta = (\theta_1, \theta_2) \tag{1.22}$$

where ν_θ is an extremal translation-invariant, invariant measure which has intensity $\theta = (\theta_1, \theta_2)$ and is ergodic.

The next class of examples are catalyst-reactant systems of the branching type, i.e. we are concerned with the case of components of X_t with values in $I = [0, \infty)$. Again we consider the case $S = \mathbb{Z}^d$. The medium is now a branching random walk with irreducible random walk q–matrix $\mathcal{B}$.

A branching random walk (BRW) is a particle system on $\mathbb{Z}^d$, i.e. a Markov process on $\mathbb{E}\cap\mathbb{N}_0^{\mathbb{Z}^d}$ evolving according to the following rules:

- Particles migrate independent of each other according to $\mathcal{B}$.
- Every particle has an exponential life time, independent of those other particles. At the end of its life time the particle is either removed or replaced by a random number of new particles at that site.
- Migration and branching occur independently of each other.

As an initial state for the bivariate evolution $(H_t, X_t)_{t\geq 0}$ we choose $\mathcal{H}(\theta_1) \otimes \delta_{\theta_2\mathbb{1}}$, where $\mathcal{H}(\theta)$ is a Poisson system with intensity θ. Now we have constructed a well-defined catalyst-reactant system and we can study its longtime behavior. Again as in the Fisher-Wright model this will be highly dimension dependent.

Both BRW and interacting Feller's branching diffusions ($g_{i,t} = g_B^c$) have the property that they become locally extinct in $d = 1, 2$ (more generally, if the symmetrized random walk kernel is recurrent) while for $d \geq 3$ (transient symmetrized random walk) an equilibrium state is approached which is translation invariant, shift ergodic and has the same intensity as the initial state. Again the behavior is different in random medium.

We prove that in one-dimensional situations the migration is the strongest force. It produces in the longtime limit constant states with preserved mass for the reactant X. This contrasts with the local extinction which occurs in the case of constant branching rates.

Theorem 6 *Assume that $d = 1$ and $\mathcal{A} = \mathcal{B}$ generate simple symmetric random walk. Further assume that $g : [0, \infty) \to \mathbb{R}^+$ is locally Lipschitz and there is a constant C such that*

$$g(x) \leq Cx, \qquad x \in [0, \infty). \tag{1.23}$$

Then

$$\mathcal{L}_{\mathcal{H}(\theta_1)\otimes\delta_{\theta_2\mathbb{1}}}\left[(H_t, X_t)\right] \overset{t\to\infty}{\Longrightarrow} \delta_{(0,\theta_2\mathbb{1})}. \tag{1.24}$$

(Note that again $\Longrightarrow$ denotes weak convergence w.r.t. the product topology in the state spaces of H and X. Also, due to spatial homogeneity, weak convergence is for the used initial states equivalent to weak convergence w.r.t. the topology induced by the Liggett-Spitzer norm. This is, of course, true also in the next theorem.)

In the case of $d = 2$ we obtain limiting states for the law of the reactant $\mathcal{L}[X_t]$ as $t \to \infty$ which are as in $d = 1$ concentrated on constant states but now the constant is random.

Theorem 7 *Assume that $d = 2$ and $\mathcal{A} = \mathcal{B}$ generate simple symmetric random walk. Further assume that $g : [0, \infty) \to \mathbb{R}^+$ is locally Lipschitz and there are constants $c, C > 0$ such that*

$$cx \le g(x) \le Cx, \qquad x \in [0, \infty). \tag{1.25}$$

Then any weak limit point $\nu(d(\eta, \xi))$ of $\mathcal{L}_{\mathcal{H}(\theta_1)\otimes\delta_{\theta_2 \mathbb{1}}}[(H_t, X_t)]$ as $t \to \infty$ is concentrated on states with

$$\eta \equiv 0 \quad \text{and} \quad \xi(i) = \xi(j),\ i, j \in S. \tag{1.26}$$

Furthermore $\int \xi(0)\nu(d(\eta, \xi)) = \theta_2$ and $\int \xi(0)^2\nu(d(\eta, \xi)) = \infty$.

For the case $d \ge 3$ (or $\mathcal{A}$, $\mathcal{B}$ transient after symmetrization) we have again translation invariant, shift-ergodic extremal equilibrium states which have the same intensity as the initial state.

2 Existence, Uniqueness and Comparison

2.1 Proof of Theorem 1

First we remark that *uniqueness* of the solution of (1.2) is proved with a Gronwall argument like in [SS80], Thm. 3.2, but now applied to $\mathbf{E}[\langle \gamma, |X_t - X_t'|\rangle]$ for two solutions X and X' of (1.2) with γ from (1.4).

The next point is to show *existence*. For that purpose we follow the classical route and consider first systems indexed by finite sets and then we pass to the limit.

Step 1. For a finite set $\Lambda \subseteq S$ and $i, j \in \Lambda$, we put $\mathcal{A}^\Lambda(i, j) := \mathcal{A}(i, j)$. Moreover, for a fixed $x_0 \in \mathbb{E}$, let x_0^Λ denote its restriction to I^Λ. Consider the finite-dimensional system

$$dX_t^\Lambda(i) = \mathcal{A}^\Lambda X_t^\Lambda(i)dt + \sqrt{2g_{i,t}(X_t^\Lambda(i))}dW_t(i), \quad i \in \Lambda; \quad X_0^\Lambda = x_0^\Lambda. \tag{2.1}$$

Combining Thm. V 20.1 and Thm. V 23.5 of [RW87] we conclude that a (weak) solution X^Λ of (2.1) exists (note that the boundedness assumption in [RW87] can be met by a stopping argument). The same reasoning as in the proof of [SS80], Thm. 3.1 and Thm. 3.2, shows that X_t^Λ remains in I^Λ a.s. for all t, and that in fact X^Λ is the *unique strong* solution of (2.1) (where we use a suitable stopping argument to meet the boundedness assumptions on the diffusion coefficients in [SS80], namely Assumption [B-1] and [B-1]').

Step 2. In order to consider sequences of processes corresponding to different sets Λ and in order to compare them we use in the sequel (2.1) for different sets Λ, but driven by the same sequence of independent Wiener processes $W(i), i \in S$.

We claim that for $\Lambda \subseteq \Lambda' \subseteq S$, Λ' finite, the following holds

$$X_t^\Lambda(i) \le X_t^{\Lambda'}(i) \quad \text{a.s.}, \qquad i \in \Lambda,\ t \ge 0. \tag{2.2}$$

Indeed, using Le Gall's local time technique for proving the Ikeda-Watanabe comparison result and proceeding similarly as in the proof of Thm. V 43.1 in [RW87] we arrive at

$$\begin{aligned} 0 &\le \mathbf{E}\int_0^t \mathbb{1}_{\{X_s^\Lambda(i)-X_s^{\Lambda'}(i)>0\}}(\mathcal{A}^\Lambda X_s^\Lambda(i)-\mathcal{A}^{\Lambda'}X_s^{\Lambda'}(i))\,ds \\ &\le \mathbf{E}\int_0^t \mathbb{1}_{\{X_s^\Lambda(i)-X_s^{\Lambda'}(i)>0\}}\mathcal{A}^\Lambda(X_s^\Lambda - X_s^{\Lambda'})(i)\,ds \\ &= \mathbf{E}\int_0^t \mathcal{A}^\Lambda(i,i)(X_s^\Lambda(i)-X_s^{\Lambda'}(i))_+ + \sum_{j\neq i}\mathcal{A}^\Lambda(i,j)(X_s^\Lambda(j)-X_s^{\Lambda'}(j))\,ds \\ &\le \mathbf{E}\int_0^t \mathcal{A}^\Lambda((X_s^\Lambda - X_s^{\Lambda'})_+)(i)\,ds. \end{aligned} \tag{2.3}$$

Note that in the last inequality we made use of the fact that $\mathcal{A}^\Lambda(i,j)\ge 0$ if $i\neq j$. Hence,

$$0\le \mathbf{E}\sum_{i\in\Lambda}(X_t^\Lambda(i)-X_t^{\Lambda'}(i))_+ \le |\Lambda|\int_0^t \mathbf{E}\left[\sum_{i\in\Lambda}(X_s^\Lambda(i)-X_s^{\Lambda'}(i))_+\right]ds \quad .$$

Using Gronwall's lemma this shows (2.2).

Thus there exists the monotone limit X_t

$$X_t^\Lambda(i)\uparrow X_t(i) \quad \text{a.s.} \tag{2.4}$$

Since $\mathbf{E}[|X_t^{\Lambda'}(i)-X_t^\Lambda(i)|]=(a_t^{\Lambda'}-a_t^\Lambda)x_0(i)$, $X_t(i)$ is also the L^1–limit of X_t^Λ. Note that $a_t^\Lambda(i,j)\uparrow a_t(i,j)$ as $\Lambda\uparrow S$. Hence monotone convergence yields

$$\mathbf{E}[X_t(i)]=\lim_{\Lambda\uparrow S}\mathbf{E}[X_t^\Lambda(i)]=\lim_{\Lambda\uparrow S}\sum_{j\in\Lambda}a_t^\Lambda(i,j)x_0(j)=a_tx_0(i).$$

Thus

$$\mathbf{E}[\langle\gamma,X_t\rangle]=\langle\gamma a_t,x_0\rangle\le e^{Mt}\langle\gamma,x_0\rangle, \qquad t\ge 0, \tag{2.5}$$

Hence X_t takes values in $\mathbb{E}$.

Step 3. In the next step of the proof we will show that X is indeed a solution of (1.2). For this purpose we fix $i\in S$ and localize with the stopping times $\tau_N:=\inf\{t:\langle\gamma,X_t\rangle\ge N\}$, $N\in\mathbb{N}$. Note that X inherits the Markov property from X^Λ. (In fact, for $t>s$, X_t^Λ is a measurable function of X_s^Λ and $\sigma(W_r(i)-W_s(i),\, r\in[s,t],\, i\in\mathbb{Z}^d)$ and hence also X_t depends only on X_s and the Brownian increments between time s and t.) Hence by (2.5) the process $(e^{-tM}\langle\gamma,X_t\rangle)_{t\ge0}$ is a supermartingale, and Doob's inequality yields $\tau_N\to\infty$ as $N\to\infty$ almost surely.

Note that

$$\int_0^{t\wedge\tau_N}\mathcal{A}^\Lambda X_s^\Lambda(i)ds=\int_0^{t\wedge\tau_N}\sum_{j\in\Lambda,\,j\neq i}\mathcal{A}(i,j)X_s^\Lambda(j)ds-\int_0^{t\wedge\tau_N}X_s^\Lambda(i)ds. \tag{2.6}$$

By dominated convergence the first term on the r.h.s. of (2.6) converges, as $\Lambda\uparrow S$, almost surely to $\int_0^{t\wedge\tau_N}\sum_{j\in S,\,j\neq i}\mathcal{A}(i,j)X_s(j)ds$, whereas the second term on the r.h.s. of (2.6) converges a.s. to $\int_0^{t\wedge\tau_N}X_s(i)ds$. Overall, we have

$$\int_0^{t\wedge\tau_N}\mathcal{A}^\Lambda X_s^\Lambda(i)\,ds\longrightarrow\int_0^{t\wedge\tau_N}\mathcal{A}X_s(i)\,ds \quad \text{a.s. as } \Lambda\uparrow S. \tag{2.7}$$

On the other hand,

$$\mathbf{E}\left[\left(\int_0^{t\wedge\tau_N}\sqrt{2g_{i,s}(X_s^\Lambda(i))}\,dW_s(i)-\int_0^{t\wedge\tau_N}\sqrt{2g_{i,s}(X_s(i))}\,dW_s(i)\right)^2\right] \tag{2.8}$$
$$=\mathbf{E}\left[\int_0^{t\wedge\tau_N}\left(\sqrt{2g_{i,s}(X_s^\Lambda(i))}-\sqrt{2g_{i,s}(X_s(i))}\right)^2 ds\right]\longrightarrow 0 \quad \text{as } \Lambda\uparrow S.$$

For fixed N this shows a.s. convergence of the martingale term in (2.1) as $\Lambda \uparrow S$, at least along a suitable subsequence. Hence both terms on the r.h.s of (2.1) converge adequately as $\Lambda \uparrow S$. Now letting $N \to \infty$ shows that X is a solution of (1.2). □

2.2 Proof of Theorem 2

Our task is to generalize the result in [CFG96] to diffusion functions which depend both on the site and the time. In fact, we only show statement (a), since the proof for statement (b) is the same as provided in [CFG96] (see also Remark 1.4). For that purpose let us recall briefly the basic idea in the homogeneous case $g_{i,t} \equiv g$.

Let $(S_t^{g^l})_{t\geq 0},\ l=1,2$ be the semigroups belonging to the system of interacting diffusions with diffusion function $g^l, l=1,2$ where $g^1 \geq g^2$ and let $G^{g^l},\ l=1,2$ denote the corresponding generators. From the fact that (because of $g^1 \geq g^2$) on $\mathcal{F}$ the inequality $(G^{g^1}-G^{g^2}) \geq 0$ holds, we get, using the positivity of $S_t^{g^l}$ and the formula of partial integration for semigroups, that for $f \in \mathcal{F}$

$$S_t^{g^1}(f)-S_t^{g^2}(f)=\int_0^t S_{t-s}^{g^1}(G^{g^1}-G^{g^2})S_s^{g^2}(f)ds \geq 0, \quad f\in\mathcal{F}, \tag{2.9}$$

provided that we can prove

$$S_t^{g^2}(f)\in\mathcal{F}. \tag{2.10}$$

From these two relations one can derive the assertion using the Markov property.

The assertion (2.10) was shown in [CFG96] first for finite S and smooth g^l, i.e. $\sqrt{g^l}\in C^2$ and gave the comparison in those cases. Then one removed the smoothness requirement and then the restriction $|S|<\infty$ was removed by approximation arguments to get the general case.

Accordingly we will show that we can generalize the results of [CFG96]. First we show (2.10) for finite smooth systems in frozen media (Step 1) and then extend this to piecewise constant (in time) media (Step 2). Clearly this implies (1.14) under the restrictions.

The time-inhomegeneous medium would require in (2.9) to work with time-inhomogeneous generators. We avoid this little technicality by generalizing (1.14) rather than (2.10) when we successively drop the assumptions smoothness (Step 3), piecewise constantness (Step 4), and finiteness (Step 5).

Step 1. Let us consider a *finite* set Λ of sites, a q–matrix $\mathcal{B}$ (or $\mathcal{B}^T$) on Λ, and (site-dependent but time-independent) diffusion functions g_i fulfilling

- $\sqrt{g_i}$ is twice continuously differentiable,
- there exists a bounded interval $(a,b)\subseteq I$ such that all the g_i vanish outside (a,b). (2.11)

Consider (for fixed $z\in I^\Lambda$) the (unique strong) solution of the system

$$dX_t(i)=\mathcal{B}X_t(i)dt+\sqrt{2g_i(X_t(i))}\,dW_t(i), \qquad i\in\Lambda, \quad X_0=z. \tag{2.12}$$

Then the same reasoning as in subsections 2.1 - 2.4 of [CFG96] shows that the semigroup associated with (2.12) preserves the function cones $\mathcal{F}$ and $\mathcal{F}^\uparrow$. In fact, they used Trotter's formula to get the result from

the one for the following two systems, which are special cases of the model:
(i) $\mathcal{B} = 0$ (independent collection of diffusion processes without drift),
(ii) $g \equiv 0$ (pure deterministic system of differential equations).

Note that in (i) there is no interaction and therefore the result of [CFG96] does not depend on their assumption that $g_i = g$, $i \in S$. Furthermore (ii) has in our context exactly the same form. Hence (2.10) still holds under the two assumptions of this Step 1.

Step 2. An induction argument based on the Markov property and the preservation of $\mathcal{F}$ and $\mathcal{F}^{\uparrow}$ by the semigroup of the process shows that the comparison result stated in Step 1 extends to space-time dependent diffusion coefficients $g_{i,t}$ which are piecewise constant over time in the following sense:

$$\begin{aligned}&\text{For all } T > 0, \text{ there exists a finite partition } 0 =: t_0 < t_1 < \ldots < t_n := T,\\ &\text{and there exist } g_i^{(1)}, \ldots, g_i^{(n)} \text{ obeying (2.11), such that for all } m = 1, \ldots, n\\ &\text{and } t \in [t_{m-1}, t_m), \text{ the function } g_{i,t} \text{ coincides with } g_i^{(m)}.\end{aligned} \tag{2.13}$$

Hence again (2.10) holds. Finally the rest of the argument that for the solutions X^1, X^2 of two systems of the form (2.12) with the same $\mathcal{B}$ and z but different diffusion functions g_i^1, g_i^2 obeying $g_i^1 \geq g_i^2$ for all $i \in \Lambda$ the comparison relations (1.14) and (1.15) hold true, remains the same.

Step 3. Next, we extend the comparison result to systems of the form

$$dX_t(i) = \mathcal{B}X_t(i)dt + \sqrt{2g_{i,t}(X_t(i))}\, dW_t(i), \qquad i \in \Lambda, \quad X_0 = z. \tag{2.14}$$

where the diffusion functions satisfy (2.13) but with (2.11) replaced by the requirement that for each $m = 1, \ldots, n$

- $g_i^{(m)}$ is locally Lipschitz,
- $g_i^{(m)}(x) = 0$ for x in the boundary of I, (2.15)
- $g_i^{(m)}$ has at most quadratic growth (if I is unbounded).

This extension is accomplished by the SDE-version of the procedure given in [CFG96]. Consider smooth $g_i^{m,\ell}$ positive on (a, b), i.e. obeying (2.11), such that $g_i^{m,\ell} \to g_i$ as $m, \ell \to \infty$. Then by a coupling, arising by using the same Brownian motions, one proves as in Section 2.1 that along suitable subsequences the solutions converge. Then by the same argument as in (2.8) we get that the solutions of our equations converge as $m, \ell \to \infty$ to a strong solution and hence by strong uniqueness to the solution. Hence the comparison holds for systems as in (2.14) as well.

Step 4. An approximation procedure now extends the comparison result to systems of the form (2.14) but with space-time dependent diffusion functions $g_{i,t}$ fulfilling assumption (1.3) only (instead of (2.13), (2.15)). Indeed, consider for $T > 0$ a sequence of partitions $\mathcal{P}_n$ of $[0, T]$ whose mesh size tends to zero, put for a partition (t_k^n), say,

$$g_{i,t}^n := g_{i,t_k^n} \quad \text{if} \quad t_k^n \leq t < t_{k+1}^n. \tag{2.16}$$

Then apply (according to Step 2) the comparison result to solutions of (2.14) (with $g_{i,t}^n$ instead of $g_{i,t}$), and pass to the limit, again using tightness and uniqueness of the solution of (2.14).

Step 5. Before we pass to infinite systems we make the following observation. The comparison result from Step 3 immediately extends to systems of the form (2.1) whose kernel, respectively the transposed kernel (being the restriction of $\mathcal{A}$ to Λ), generates a sub-Markovian (instead of a Markovian) semigroup. To see this, it suffices to introduce an auxiliary site $\Delta \notin \Lambda$ and to extend the system in (2.12) to $\widetilde{\Lambda} := \Lambda \cup \{\Delta\}$ by choosing b and $g_{\Delta,t}$ as

$$\mathcal{B}(i, \Delta) := \sum_{j \in \Lambda^c} \mathcal{A}(i, j), \qquad \mathcal{B}(\Delta, j) := 0, \quad j \in \widetilde{\Lambda}, \qquad X_0^\Lambda(\Delta) := 0, \quad g_{\Delta,t} := 0. \tag{2.17}$$

To complete the proof, it remains to pass from the finite to the infinite systems, i.e. from (2.1) to (1.2). To this end, fix site- and time dependent diffusion functions $g^1_{i,t}, g^2_{i,t}$ meeting the conditions (1.3) and obeying the relation (1.11). Then above observation yields that for each finite $\Lambda \subseteq S$, the comparison result (i.e. the relations (1.14) and (1.15)) holds true for the solutions $X^{1,\Lambda}, X^{2,\Lambda}$ of two systems of the form (2.1) with kernel $\mathcal{B}$ the restriction of $\mathcal{A}$ to Λ and with the diffusion functions $g^1_{i,t}, g^2_{i,t}$, respectively. Finally, since (2.4) asserts convergence of the solutions of (2.1) towards that of (1.2) as $\Lambda \uparrow S$, the comparison relations (1.14) and (1.15) carry over from the solutions $X^{1,\Lambda}, X^{2,\Lambda}$ to the solutions X^1, X^2 of their infinite-dimensional counterparts (1.2). This proves Theorem 2. □

3 Interacting diffusions in a voter medium

Proof of Theorem 3 and 4

In the special case $g = g^1_{FW}$ the statement of Theorem 4 was shown in [GKW01, Theorem 2]. Thus we only have to proof Theorem 3 since Theorem 4 is a corollary of it and [GKW01, Theorem 2].

Let $f \in C_b^+([0,1]^{\mathbb{Z}^d})$, let $R \subset \mathbb{Z}^d$ be finite and assume that $\lambda \in [0,\infty)^{\mathbb{Z}^d}$ is such that $\lambda(i) = 0$ for $i \in \mathbb{Z}^d \setminus R$. Recall that $F_\lambda(\xi) = \exp(-\langle \lambda, \xi\rangle)$. Note that the expectations $\mathbf{E}[f(H_t)F_\lambda(X_t)]$ determine the distribution of (H_t, X_t), hence it suffices to show the convergence of these expectations. We will do this by obtaining bounds from above and below which turn out to agree. The bounds are based on the fact that from [GKW01, Theorem 2 and 3] we know that the statement is true for $g = g^c_{FW}$, $c > 0$ (recall (1.9)).

Upper bound. Fix a g and note that there exists a $c > 0$ such that $g^c_{FW} \geq g$. Define X^c as X but with g^c_{FW} instead of g. From Theorem 2 we know that

$$\mathbf{E}[f(H_t)F_\lambda(X_t)] \leq \mathbf{E}[f(H_t)F_\lambda(X^c_t)], \qquad t \geq 0. \tag{3.1}$$

Hence by [GKW01, Theorem 2 and 3]

$$\begin{aligned}\limsup_{t\to\infty} \mathbf{E}[f(H_t)F_\lambda(X_t)] &\leq \lim_{t\to\infty} \mathbf{E}[f(H_t)F_\lambda(X^c_t)] \\ &= \mathbf{E}[f(\widetilde{H}\mathbb{1})F_\lambda(\widetilde{X}\mathbb{1})].\end{aligned} \tag{3.2}$$

Lower bound. If $\theta_2 \in \{0,1\}$ then in (3.1) equality holds and we are done. We may thus assume $\theta_2 \in (0,1)$. Let $\varepsilon \in (0,1)$ and define $I^\varepsilon = [\varepsilon\theta_2, 1-\varepsilon(1-\theta_2)]$. For $c > 0$ define

$$g^{c,\varepsilon}_{FW} : [0,1] \to [0,\infty), \quad x \mapsto c\cdot(x-\varepsilon\theta_2)^+(1-\varepsilon(1-\theta_2)-x)^+.$$

Choose $c = c(\varepsilon) > 0$ such that $g^{c,\varepsilon}_{FW} \leq g$ and define $X^{c,\varepsilon}$ as X but with $g^{c,\varepsilon}_{FW}$ instead of g. By Theorem 2 we have

$$\mathbf{E}[f(H_t)F_\lambda(X_t)] \geq \mathbf{E}[f(H_t)F_\lambda(X^{c,\varepsilon}_t)], \qquad t \geq 0. \tag{3.3}$$

On the other hand $X^{c,\varepsilon}$ really lives on $(I^\varepsilon)^{\mathbb{Z}^d}$ (since $g^{c\varepsilon}_{FW}(x) = 0$, $x \in I \setminus I^\varepsilon$, and $\theta_2 \in I^\varepsilon$). Hence it is simple to check that the following scaling relation holds

$$\mathcal{L}[X^{c,\varepsilon}|H] = \mathcal{L}[\varepsilon\theta_2 + (1-\varepsilon)X^c|H]. \tag{3.4}$$

Thus we get

$$\begin{aligned}\liminf_{t\to\infty} \mathbf{E}[f(H_t)F_\lambda(X_t)] &\geq \lim_{t\to\infty} \mathbf{E}[f(H_t)F_\lambda(\varepsilon\theta_2 + (1-\varepsilon)X^c_t)] \\ &= \exp(-\varepsilon\theta_2|R|)\mathbf{E}[f(\widetilde{H}\mathbb{1})F_{(1-\varepsilon)\lambda}(\widetilde{X}\mathbb{1})] \\ &\geq \exp(-\varepsilon\theta_2|R|)\mathbf{E}[f(\widetilde{H}\mathbb{1})F_\lambda(\widetilde{X}\mathbb{1})].\end{aligned} \tag{3.5}$$

Now let $\varepsilon \to 0$ and combine this with (3.2) to obtain

$$\lim_{t\to\infty} \mathbf{E}[f(H_t)F_\lambda(X_t)] = \mathbf{E}[f(\widetilde{H}\mathbb{1})F_\lambda(\widetilde{X}\mathbb{1})] \tag{3.6}$$

as desired. □

Proof of Theorem 5

The proof of Theorem 5 works similarly as the proof of Theorem 3 (and 4) and we only give a sketch here.

Again for $g = g^c_{FW}$ the statement is proved in [GKW01]. For the general case we proceed as above. Fix $N \in \mathbb{N}$ and time points $s_1, \dots, s_N$. For each $i = 1, \dots, N$ choose a finite set $R_i = \{r_{i,1}, \dots, r_{i,n_i}\} \subset \mathbb{R}$ and numbers $\mu_{i,1}, \dots, \mu_{i,n_i} \in [0, \infty)$. For $t > 0$ define $R^t_i = \{\lfloor t^{1/2} r_{i,j} \rfloor : j = 1, \dots, n_i\}$ and $\lambda^t_i \in [0,\infty)^{R^t_i}$ by $\lambda^t_{i,\lfloor t^{1/2} r_{i,j}\rfloor} = \mu_{i,j}$. Finally define

$$F^t_i(\xi) = \exp(-\langle \xi, \lambda^t_i \rangle) = \exp\left(-\sum_{j=1}^{n_i} \mu_{i,j} \xi(\lfloor t^{1/2} r_{i,j} \rfloor) \right).$$

Similarly define functions f^t_i, $i = 1, \dots, N$ with different choices of R and μ.

Now choose $g^c_{FW} \geq g$ and define X^c as X but with g replaced by g^c_{FW}. From Theorem 2(b) we know that

$$\begin{aligned}
\mathbf{E}\left[\prod_{i=1}^N f^t_i(H_{ts_i}) F^t_i(X_{ts_i}) \right] &= \mathbf{E}\left[\prod_{i=1}^N f^t_i(H_{ts_i}) \mathbf{E}\left[\prod_{i=1}^N F^t_i(X_{ts_i}) \big| H \right] \right] \\
&\leq \mathbf{E}\left[\prod_{i=1}^N f^t_i(H_{ts_i}) \mathbf{E}\left[\prod_{i=1}^N F^t_i(X^c_{ts_i}) \big| H \right] \right] \\
&= \mathbf{E}\left[\prod_{i=1}^N f^t_i(H_{ts_i}) F^t_i(X_{ts_i}) \right].
\end{aligned}$$

This yields an upper bound as in the proof of Theorems 3 and 4. For the lower bound use functions $g^{c,\varepsilon}_{FW}$ and again proceed as above. We omit further details. $\square$

4 Catalyst-reactant systems of the branching type

4.1 Preparation: Some facts on branching systems

Recall that here the medium H is branching random walk. Consider the solution X^c of (1.2) with $g_{i,t} = H_t(i) g^c_B$, where $g^c_B(x) = c \cdot x$, $x \geq 0$. In order to show Theorem 6 and 7 one would like to use such an X^c as a reference system. Thus we need to show the statements for X^c first and then use the comparison theorem.

In the literature the corresponding statements have been shown for catalytic branching random walk $(\xi^c_t)_{t \geq 0}$ instead of X^c. In this process, particles perform independent random walks (with the transposed kernel $\mathcal{A}^T$) and die with rate $cH_t(i)$. At the place of death either no or two new particles are created, each possibility occurring with probability $\frac{1}{2}$. This model has been investigated in [GKW99] in quite some detail. It was shown that in $d = 1$ and with $\mathcal{A}$ and $\mathcal{B}$ symmetric simple random walk

$$\mathcal{L}_{\mathcal{H}(\theta_1) \otimes \mathcal{H}(\theta_2)}[(H_t, \xi^c_t)] \overset{t \to \infty}{\Longrightarrow} \delta_0 \otimes \mathcal{H}(\theta_2). \tag{4.1}$$

In $d = 2$ (and again $\mathcal{A}$ and $\mathcal{B}$ symmetric simple random walk)

$$\mathcal{L}_{\mathcal{H}(\theta_1) \otimes \mathcal{H}(\theta_2)}[(H_t, \xi^c_t)] \overset{t \to \infty}{\Longrightarrow} \delta_0 \otimes \mathbf{E}[\mathcal{H}(\zeta^c)], \tag{4.2}$$

where ζ^c is a random variable with $\mathbf{E}[\zeta^c] = \theta_2$ and $\mathbf{Var}[\zeta^c] = \infty$. More precisely, ζ^c can be represented as the density of catalytic super Brownian motion (in $d = 2$) at time c at the origin, say (see also [FK99] or [Kle00]).

Now there is a simple connection between ξ^c and X^c: roughly speaking we obtain ξ^c_t from X^c_t as a Poisson point process on $\mathbb{Z}^d$ with intensity X^c_t. More precisely, (see, e.g., [Kle98, (4.19)] or [Kle01])

$$\mathcal{L}_{\mathcal{H}(\theta_1) \otimes \delta_{\theta_2 \mathbb{1}}}[(H_t, \mathcal{H}(X^c_t))] = \mathcal{L}_{\mathcal{H}(\theta_1) \otimes \mathcal{H}(\theta_2)}[(H_t, \xi^c_t)]. \tag{4.3}$$

Thus (4.1) and (4.2) translate into

$$\mathcal{L}_{\mathcal{H}(\theta_1),\theta_2\mathbb{1}}[(H_t, X_t^c)] \overset{t\to\infty}{\Longrightarrow} \begin{cases} \delta_0 \otimes \delta_{\theta_2\mathbb{1}}, & d=1, \\ \delta_0 \otimes \mathbf{E}[\delta_{\zeta^c\mathbb{1}}], & d=2. \end{cases} \tag{4.4}$$

where ζ^c is a random variable with $\mathbf{E}[\zeta^c] = \theta_2$ and $\mathbf{E}[\zeta^c]^2 = \infty$.

4.2 Proof of Theorem 6

Recall the notation from the last subsection. Fix a function g and $C > 0$ such that $g \le g_B^C$. By (4.4) for $i \in S$ and $\lambda \ge 0$

$$\lim_{t\to\infty} \mathbf{E}[\exp(-\lambda X_t^C(i))] = \exp(-\lambda\theta_2). \tag{4.5}$$

Thus by the comparison theorem

$$\limsup_{t\to\infty} \mathbf{E}[\exp(-\lambda X_t(i))] \le \exp(-\lambda\theta_2). \tag{4.6}$$

However, for any weak limit point ν of $\mathcal{L}[X_t]$ as $t \to \infty$ we have $\vartheta := \int x\, \nu(dx) \le \theta_2$. Jensen's inequality now yields

$$\int \exp(-\lambda x)\, \nu(dx) \ge \exp(-\lambda\vartheta) \ge \exp(-\lambda\theta_2).$$

Together with (4.6) this implies

$$\lim_{t\to\infty} \mathbf{E}[\exp(-\lambda X_t(i))] = \exp(-\lambda\theta_2), \tag{4.7}$$

and hence the theorem is proved. □

4.3 Proof of Theorem 7

First of all recall from (4.4) that for $g = g_B^c$

$$\mathcal{L}_{\mathcal{H}(\theta_1)\otimes\delta_{\theta_2\mathbb{1}}}[(H_t, X_t^c)] \overset{t\to\infty}{\Longrightarrow} \delta_0 \otimes \mathbf{E}[\delta_{\zeta^c\mathbb{1}}], \tag{4.8}$$

and that $\mathbf{E}[\zeta^c] = \theta_2$, $\mathbf{Var}[\zeta^c] = \infty$. In other words, for finite $R \subset \mathbb{Z}^2$ and $\lambda, \lambda' \in [0,\infty)^R$

$$\lim_{t\to\infty} \mathbf{E}\big[e^{-\langle\lambda', H_t\rangle - \langle\lambda, X_t^c\rangle}\big] = \mathbf{E}\big[e^{-\langle\lambda,\mathbb{1}\rangle\zeta^c}\big]. \tag{4.9}$$

Furthermore, for $f_c(\rho) := \mathbf{E}[e^{-\rho\zeta^c}]$, $\rho \ge 0$, the first and second derivatives are

$$f_c'(0) = -\theta_2, \qquad f_c''(0) = \infty. \tag{4.10}$$

Now we come back to the general situation where $c > 0$ and $C > 0$ are such that $g_c \le g \le g_C$. In a first step we consider only one coordinate. In the second step we show that in the longrun all coordinates are close with high probability.

Step 1 (One Coordinate).
Let μ be a weak limit point of $\mathcal{L}_{\mathcal{H}(\theta_1)\otimes\delta_{\theta_2\mathbb{1}}}[X_t]$. By the comparison theorem for λ as above

$$f_C(\langle\lambda,\mathbb{1}\rangle) \le \int \mu(d\xi)\exp(-\langle\lambda,\xi\rangle) \le f_c(\langle\lambda,\mathbb{1}\rangle). \tag{4.11}$$

In particular, for $f(\rho) := \int \mu(d\xi)\exp(-\rho\xi(0))$, we have $f'(0) = -\theta_2$ and $f''(0) = \infty$. Hence

$$\int \mu(d\xi)\xi(0) = \theta_2, \qquad \int \mu(d\xi)\xi^2 = \infty. \tag{4.12}$$

We are done if we can show that μ–almost surely $\xi(i)=\xi(j)$. Note that (4.11) alone is not sufficient to show that the coordinates of ξ all agree. In fact, there are very simple counterexamples. Thus we have to rely on a different argument.

Step 2 (The Coordinates Agree in the Longrun).
Fix two sites $i_1,i_2\in\mathbb{Z}^2$ and $\varepsilon>0$. We want to show that for T large enough $\mathbf{P}[|X_T(i_1)-X_T(i_2)|>\varepsilon]<\varepsilon$. The idea is that for large T the i_1 and i_2 have experienced for a longtime vanishing branching rate and hence become equal by the mass flow.

Let $S>0$ be large enough that

$$\sum_{k\in\mathbb{Z}^2}|a_S(i_1,k)-a_S(i_2,k)|\le\frac{\varepsilon^2}{12\theta_2 C}. \tag{4.13}$$

Hence

$$\begin{aligned}\mathbf{P}\Big[|a_SX_{T-S}(i_1)-a_SX_{T-S}(i_2)|>\frac{\varepsilon}{3}\Big]&\le\frac{3}{\varepsilon}\mathbf{E}\big[|a_SX_{T-S}(i_1)-a_SX_{T-S}(i_2)|\big]\\&\le\frac{\varepsilon}{4\theta_2}\mathbf{E}[X_{T-S}(0)]\\&=\frac{\varepsilon}{4}.\end{aligned} \tag{4.14}$$

Fix $R>0$ such that

$$\int_0^S du\sum_{\|k\|>R}a_u(i_1,k)^2+a_u(i_2,k)^2<\frac{\varepsilon^3}{36\theta_2 C}. \tag{4.15}$$

For $T>0$ define the event

$$A_{S,R}(T)=\big\{H_t(k)=0,\ t\in[T-S,T],\ \|k\|\le R\big\}. \tag{4.16}$$

From [GKW99, Proposition 1.5] we know that $\mathbf{P}[A_{S,R}(T)]\overset{T\to\infty}{\longrightarrow}1$. Hence we may assume that T_ε is such that

$$\sup\big\{\mathbf{P}[A_{S,R}(T)^c],\ T\ge T_\varepsilon\big\}<\frac{\varepsilon}{4}. \tag{4.17}$$

Note that for any $i\in\mathbb{Z}^2$

$$\mathbf{E}[(X_T(i)-a_SX_{T-s}(i))^2|H]=\int_0^S du\sum_{k\in\mathbb{Z}^2}a_u(i,k)^2\mathbf{E}[g(X_{T-u}(k))|H]. \tag{4.18}$$

Combining this with (4.16) and (4.15) and using the assumption $g(x)\le Cx$, $x\ge 0$, and that $\mathbf{E}[X_r(k)]=\theta_2$, $r\ge0$, $k\in\mathbb{Z}^2$, we get for $l=1,2$

$$\begin{aligned}\mathbf{E}\big[(X_T(i_l)-a_SX_{T-S}(i_l))^2;A_{S,R}(T)\big]&\le\int_0^S du\sum_{\|k\|>R}a_u(i_l,k)^2\mathbf{E}[g(X_{T-u}(k))]\\&\le\frac{\varepsilon^3}{36}.\end{aligned} \tag{4.19}$$

This yields

$$\mathbf{P}\big[|X_T(i_l)-a_SX_{T-S}(i_l)|>\varepsilon/3;\ A_{S,R}(T)\big]\le\frac{\varepsilon}{4}. \tag{4.20}$$

Thus for all $T>T_\varepsilon$

$$\begin{aligned}\mathbf{P}[|X_T(i_1)-X_T(i_2)|>\varepsilon]&\le\mathbf{P}\big[|a_SX_{T-S}(i_1)-a_SX_{T-s}(i_2)|>\frac{\varepsilon}{3}\big]\\&\quad+\sum_{l=1}^{2}\mathbf{P}\Big[|X_T(i_l)-a_SX_{T-S}(i_l)|>\frac{\varepsilon}{3};A_{S,R}(T)\Big]+\mathbf{P}[A_{S,R}(T)^c]\\&\le\frac{\varepsilon}{4}+2\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.\end{aligned} \tag{4.21}$$

This concludes the proof of Theorem 7. □

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