ABSOLUTE CONTINUITY OF CATALYTIC MEASURE–VALUED BRANCHING PROCESSES

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Abstract. Classical super–Brownian motion (SBM) is known to take values in the space of absolutely continuous measures only if $d = 1$. For $d \geq 2$ its values are almost surely singular with respect to Lebesgue measure. This result has been generalized to more general motion laws and branching laws (yielding different critical dimensions) and also to catalytic SBM.

In this paper we study the case of a catalytic measure–valued branching process in $\mathbb{R}^d$ with a Feller process ξ as motion process, where the branching rate is given by a continuous additive functional of ξ, and where also the (critical) branching law may vary in space and time.

We provide a simple sufficient condition for absolute continuity of the values of this process. This criterion is sharp for the classical cases. As a partial converse we also give a sufficient condition for singularity of the states.

1. Introduction

1.1. **Motivation.** Classical super–Brownian motion (SBM) is a (time–homogeneous) Markov process that takes values in the space $\mathcal{M}_f(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$. It arises as the high–density short–lifetime (diffusive) limit of critical binary branching Brownian motion. There is a vast literature on this issue and we only briefly refer to Dawson (1993) for an overview.

A fundamental question is whether the states X_t are absolutely continuous with respect to Lebesgue measure ℓ or if they are singular. It is well known for classical SBM (see Dawson and Hochberg (1979)) that for fixed positive time, $X_t \ll \ell$ almost surely if $d < 2$ and $X_t \perp \ell$ if $d \geq 2$. The aim of this paper is to give a sufficient condition for absolute continuity for a broader class of measure–valued spatial branching processes.

1.2. **Earlier Results.** The model of classical SBM allows for some generalizations. In order to describe these generalizations properly we have to give a more detailed description of SBM first (see [Daw93] for a more intense treatment). Let $s \in \mathbb{R}$ and $\mu \in \mathcal{M}_f(\mathbb{R}^d)$. Denote by $\mathbf{P}_{s,\mu}$ (respectively $\mathbf{E}_{s,\mu}$) the probability distribution of (respectively the expectation with respect to) the process $(X_t)_{t\geq s}$ when started at time s in μ. For fixed $x \in \mathbb{R}^d$, $t > s$ and a test function $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ (the space of bounded non–negative Borel functions

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on $\mathbb{R}^d$) define the log–Laplace transform

$$u(s,t,x;\ f) = -\log(\mathbf{E}_{s,\delta_x}[\exp(-\langle X_t, f\rangle)]). \tag{1.1}$$

(Note that $u(s,t,x;\ f) = u(0,t-s,x;\ f)$, however we use the inhomogeneous notation in order to prepare for a time–inhomogeneous situation) Here δ_x denotes the Dirac measure in x and $\langle \mu, f\rangle := \int f\ d\mu$. By the branching property we have multiplicativity of the process (X_t), that is

$$-\log(\mathbf{E}_{s,\mu}[\exp(-\langle X_t, f\rangle)]) = \langle \mu, u(s,t,\ \cdot\ ;\ f)\rangle. \tag{1.2}$$

In particular, the knowledge of the initial measure and of u determines the law of X. The function u is the (unique non–negative) solution of a simple semi–parabolic partial differential equation (or *reaction–diffusion equation*)

$$\begin{aligned} -\frac{d}{ds}u(s,t,x;\ f) &= \frac{1}{2}\Delta u - u^2, \qquad s<t,\\ u(t,t,x;\ f) &= f(x). \end{aligned} \tag{1.3}$$

We can rewrite (1.3) as the integral equation

$$u(s,t,x;\ f) = P_{t-s}f(x) - \mathbf{E}_{s,x}\left[\int_s^t u(r,t,W_r;\ f)^2\ dr\right], \qquad s\le t, \tag{1.4}$$

where $\mathbf{E}_{s,x}$ denotes the expectation with respect to the Brownian motion W that is started at time s in x and (P_r) denotes the family of heat kernels on $\mathbb{R}^d$.

Essentially three generalizations have been studied:

Motion process. Instead of Brownian motion (W_t) as spatial process for the "infinitesimal particles" one could consider any Feller process (ξ_t) in $\mathbb{R}^d$. This process might even be time–inhomogeneous. In this case, one has to replace in (1.4) the homogeneous kernel P_{t-s} by a $P_{s,t}$. Of course, the question of absolute continuity of X_T does not make sense if the transition probabilities do not have densities.

Special attention has been paid to the case of a spherically symmetric α–stable motion process ($\alpha\in(0,2]$), that is, with generator $\Delta_\alpha := -(-\Delta)^{\alpha/2}$.

Branching law. Instead of critical binary branching for the approximating branching particle system one could consider more general offspring laws. In (1.3) we have to replace the *reaction term* u^2 by $\psi(u)$ where $\psi:[0,\infty)\to[0,\infty)$ is the log–Laplace transform of a centered infinitely divisible random variable. That is, ψ has a Lévy–Khintchine representation

$$\psi(u) = au^2 + \int_0^\infty (e^{-zu} - 1 - zu)\mathbf{n}(dz), \tag{1.5}$$

where $a\ge 0$ and $\mathbf{n}$ is a measure on $(0,\infty)$ with $\int(z\wedge z^2)\mathbf{n}(dz)<\infty$. (Note that we could add another term bu for some $b\in\mathbb{R}$ which leads to non–critical branching. However this does not change the absolute continuity properties of the model. Hence for simplicity we stick to critical branching.)

Special attention has been paid to the case where $\psi(u) = u^{1+\beta}$ for some $\beta\in(0,1]$. The corresponding offspring distribution has moments of order smaller than $1+\beta$ only. It occurs, for example, as the limit of the branching particle system with offspring distribution $(p_n)_{n\in\mathbb{N}_0}$ given by $p_0 = 1/2$, $p_1 = (1-\beta)/2$, and $p_n = \frac{1}{2}(-1)^n\binom{1+\beta}{n}$ for $n=2,3,\ldots$.

Fleischmann showed (see the appendix of [Fle88]) that in the case of a motion with generator Δ_α and with $\psi(u) = u^{1+\beta}$ the states of X_t are almost surely absolutely continuous if $d < \alpha/\beta$. On the other hand the states are almost surely singular (given non–extinction) if $d \geq \alpha/\beta$. For $d = \alpha/\beta$ this follows from the self–similarity of the process (see, for example, the appendix of [Fle88]). For $d > \alpha/\beta$ it is a simple consequence of the deeper result that the carrying dimension of the states is almost surely (given non–extinction) equal to α/β (see Section 7.2 of Dawson (1992)).

Branching rate. Instead of changing the branching *law* one can also change the branching *rate*. That is, instead of u^2 in (1.3) write $\varrho(s,x)u^2$, where the non–negative function ϱ is the branching rate. More generally, one could replace the function ϱ by a measure in time and space. The suitable way to do so is to take a continuous additive functional $A(dr)$ of Brownian motion W_r. (If A is absolutely continuous it can be written in the form $A(dr) = \varrho(r, W_r)dr$.) We define u as the solution of

$$u(s,t,x;\ f) = P_{t-s}f(x) - \mathbf{E}_{s,x}\left[\int_s^t u(r,t,W_r;\ f)^2\ A(dr)\right]. \tag{1.6}$$

The reader has to be warned that this generalization does not work for all A but one has to make strong assumptions on A in order that there exists a SBM with log-Laplace transforms given by (1.6). See, e.g., Dynkin (1991) or Fleischmann and Klenke (1999).

Let us give a short overview over the literature dealing with the question of absolute continuity of the so–called catalytic SBM (CSBM). (A survey with a broader scope can be found in [Kle99].) Delmas (1996) considers the case where A is time–homogeneous, that is, A is the collision local time of Brownian motion with a measure ν on $\mathbb{R}^d$, the so–called Revuz measure. Usually ν is considered as the distribution of a mass that catalyzes the branching. Delmas gives a capacity–type condition on ν such that X is well–defined. Furthermore he shows that off the support of ν, X_t has a smooth density that solves the heat equation.

Of special interest has also been the case where A is the collision local time of Brownian motion with a second (autonomous) super-Brownian motion. For $d = 1$, Dawson and Fleischmann (1997) show absolute continuity. The more surprising result of absolute continuity in this model even for $d = 2, 3$ can be found in [FK99]. (For $d \geq 4$ the process is trivial: the reactant is just the heat flow.) Here it is used that the catalyst itself lives on such a thin set that there is enough smoothing to obtain absolute continuity. In [FK00] it is shown that in dimension $d = 3$ the density is (given non–extinction) strictly positive almost everywhere. This is not the case in $d = 1$ and is an open problem for $d = 2$.

In [DF95] absolute continuity is shown for somewhat more general branching functionals. However, in that article very restrictive moment assumptions on A are made (see their Definition 2.4.7). A major goal of this paper is to relax these conditions. Finally, we like to mention that absolute continuity of CSBM was considered for a very special class of catalysts in [DFR91].

1.3. **Our Model.** As the motion process we consider a (possibly) time–inhomogeneous Feller process $(\xi_t)_{t\in[L,T]}$ in $\mathbb{R}^d$ during the fixed time interval $[L,T] \subset \mathbb{R}$ for some $L < T$. We denote its transition kernels by $P_{s,t}$, $s < t$, and assume that $P_{s,T}(x, dy)$ has a density

$p_{s,T}(x,y)$ for all $s < T$. This technical requirement is not really severe. It is met, for example, by Brownian motion, strictly elliptic diffusions, and Lévy processes with infinite Lévy measure in all coordinates (such as spherically symmetric stable processes). This assumption allows us to define for a finite measure ν on $\mathbb{R}^d$ the function $p_{s,T}\nu(x) = \langle \nu, p_{s,T}(x, \cdot\,)\rangle$. Clearly, for ν with density f, we have $p_{s,T}\nu = P_{s,T}f$.

We also allow a general branching rate as well as a general branching law that might even be time–space dependent. More precisely, we assume that $(s,x) \mapsto \psi(s,x;\ \cdot\ )$ is measurable and that for fixed s and x, the map $\psi(s,x;\ \cdot\ ) : [0,\infty) \to [0,\infty)$ is the log-Laplace transform of a centered infinitely divisible random variable. That is, we assume that ψ can be written in the Lévy–Khintchine form

$$\psi(t,x;\ u) = a(t,x)u^2 + \int_0^\infty (e^{-zu} - 1 + zu)\mathbf{n}(t,x;\ dz). \tag{1.7}$$

Here we assume that a is measurable, bounded and non–negative and that $\mathbf{n}$ is a kernel such that $(t,x) \mapsto \int_0^\infty (z \wedge z^2)\mathbf{n}(t,x;\ dz)$ is bounded.

A is a continuous non–negative additive functional of ξ. We also assume that for $x \in \mathbb{R}^d$

$$\mathbf{E}_{L,x}\left[\int_L^T A(ds)\right] < \infty. \tag{1.8}$$

Suppose that A is a branching rate functional for the function ψ. That is, we assume that there exists a multiplicative measure–valued (time–inhomogeneous) Markov process X such that (for every test function $f \in \mathcal{B}_b^+(\mathbb{R}^d)$) its log–Laplace functionals u_A (defined as in (1.6)) solve the equation

$$u_A(s,t,x;\ f) = P_{s,t}f(x) - \mathbf{E}_{s,x}\Big[\int_s^t A(dr)\psi(r,\xi_r;\ u_A(r,t,\xi_r;\ f))\Big], \tag{1.9}$$

for all $x \in \mathbb{R}^d$ and $s,t \in [L,T]$ with $s \le t$. Finally, we assume that the solution of (1.9) is unique for any $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ and $t \in [L,T]$. Clearly $u_A(s,t,\ \cdot\ ;\ f) \in \mathcal{B}_b^+(\mathbb{R}^d)$. Note that u_A has the semigroup property

$$u_A(s,t,x;\ f) = u_A(s,r,x;\ u_A(r,t,\ \cdot\ ;\ f)), \qquad L \le s \le r \le t \le T. \tag{1.10}$$

In fact, if we define for fixed t the function $v(s,x) = u(s,t,x;\ f)$, then by writing $\int_s^t = \int_s^r + \int_r^t$ we see that

$$v(s,x) = P_{s,r}v(r,\ \cdot\ )(x) - \mathbf{E}_{s,x}\Big[\int_s^r A(dr')\psi(r',\xi_{r'};\ v(r',\xi_{r'}))\Big]. \tag{1.11}$$

On account of the uniqueness assumption we have $v(s,x) = u(s,r,x;\ u(r,t,\ \cdot\ ;\ f))$. Note that we have not used the uniqueness at time t but only at time r. Hence we also have the semigroup property even if we do not have uniqueness of solutions of (1.9) with $t = T$ *a priori*. In particular, if we replace f by a measure ν, then *any* solution of (1.9) has the semigroup property.

1.4. **Results.** A first step towards checking absolute continuity is to determine whether in (1.9) we can replace the test function f by a finite measure. Brézis and Friedman show (see [BF83]) for $\xi = W$, $A(ds) = ds$ and $\psi(u) = u^\gamma$, $\gamma > 0$, that equation (1.9) has a solution with f replaced by δ_0 if and only if $\gamma < 2/d$. Hence it is clear that we will need extra conditions.

Definition 1.1. *A measure $\nu \in \mathcal{M}_f(\mathbb{R}^d)$ is called* regular *if for all $r \in [L,T)$ the map $x \mapsto p_{r,T}\nu(x)$ is bounded and for almost all $x \in \mathbb{R}^d$*

$$\mathbf{E}_{L,x}\left[\int_L^T A(dr)\psi(r,\xi_r;\ p_{r,T}\nu(\xi_r))\right] < \infty. \tag{1.12}$$

We state the following intermediate result that is of some analytical interest in its own.

Proposition 1.2. *Assume that ν is a regular measure. Then for every $\lambda \leq 1$ there exists exactly one solution of* (1.9) *with f replaced by $\lambda\nu$. This solution has the property*

$$\lim_{\lambda\downarrow 0} \lambda^{-1} u_A(s,T,x;\ \lambda\nu) = p_{s,T}\nu(x). \tag{1.13}$$

It is quite clear that if δ_y is regular for almost all $y \in \mathbb{R}^d$, the statement of Proposition 1.2 is pretty close to yielding absolutely continuous states. Though for technical reasons we need a slightly stronger condition here.

Assumption 1.3. *Assume that for almost all $y \in \mathbb{R}^d$ the measure δ_y is regular. Further let $\mu \in \mathcal{M}_f(\mathbb{R}^d)$ fulfill the assumption: There exists a bounded continuous function $\varphi : \mathbb{R}^d \to [0,\infty)$ with $\int \varphi(x)\,dx = 1$ such that for $\mu \otimes \ell$–almost all (x,y)*

$$\lim_{s\uparrow T} \limsup_{\gamma\downarrow 0} \mathbf{E}_{L,x}\Big[\int_s^T A(dr)\psi(r,\xi_r;\ P_{r,T}(\tau_y\varphi_\gamma)(\xi_r))\Big] = 0. \tag{1.14}$$

where $\varphi_\gamma(z) := \gamma^{-d}\varphi(\gamma^{-1}z)$ and τ_y denotes the shift by y.

Notice that this assumption implies in particular that the densities $p_{s,T}(x,y)$ are bounded as functions of x. Note also that a simple application of Jensen's inequality shows that (1.14) holds if the left hand side in (1.12) is bounded in y with $\nu = \delta_y$.

Now we come to formulate the main result of this paper. Recall that $[L,T]$ is the fixed time interval in which X lives.

Theorem 1. *Under the Assumption 1.3, the random measure X_T is $\mathbf{P}_{L,\mu}$–almost surely absolutely continuous w.r.t. Lebesgue measure.*

Remark Our Assumption 1.3 is really a less restrictive condition than the one of [DF95], Definition 2.4.7, where, essentially, in addition to our condition that the first moment vanishes, it is assumed that all moments vanish.

It is not hard to check that for the cases considered in Section 1.2 the conditions for absolute continuity are equivalent to (1.14). In this sense (1.14) is sharp. Clearly, one cannot expect an "if and only if" statement here, since due to the non–homogeneity of the problem we may have absolute continuity in one part of the space and singularity in another part. However, we can formulate a partial converse. Recall that $p_{s,T}(x,y)$ is the transition density of ξ_T.

Theorem 2. *Suppose that $\mu \in \mathcal{M}_f(\mathbb{R}^d)$ and that for $\mu \otimes \ell$–almost all (x,y) the solutions $u_A(L,T-\varepsilon,x;\ p_{T-\varepsilon,T}(\,\cdot\,,y))$ of* (1.9) *vanish as $\varepsilon \to 0$. Then $\mathbf{P}_{L,\mu}$–almost surely, X_T is carried by a Lebesgue null set.*

Our condition in Theorem 2 implies non–existence of a solution of (1.9) with a Dirac measure as terminal condition. The condition is met, for example, in the case where $\xi = W$, $A(ds) = ds$, $\psi(u) = u^{1+\beta}$ if and only if $\beta \geq 2/d$ (see [BF83, Theorem 2]). This brings back the classical result that super–Brownian motion with branching law determined by $\psi(u) = u^{1+\beta}$ has states that are almost surely singular to Lebesgue measure in dimension $d \geq 2/\beta$.

Of course, every generalization of a theorem asks for new examples that justify the effort. Our focus, however, lies more on the simplification of the conditions and the proofs. The reader is invited to think about interesting new examples.

1.5. **Techniques and Outline.** In earlier papers absolute continuity has been proved using moment computations. (In fact, these computations were also used to construct the processes and to investigate their path properties.) This approach forced to make strong assumptions on the moments of A. In our proofs the main tool is the maximum principle for the solutions of equation (1.9). We only have to consider the first moment. This allows us to relax the assumptions that were made on A considerably and to shorten the proof to a minimum.

In the next section we recall the maximum principle and then prove Proposition 1.2 and Theorem 1 and 2.

2. Proofs

The main ingredient to the proofs to come is the *maximum principle* for the solutions of (1.9). Define $A_\varepsilon(dr) = \mathbf{1}_{[L,T-\varepsilon]}(r)A(dr)$ for $\varepsilon \geq 0$. Note that A_ε is the branching rate functional of the super–process X^ε derived from X by switching off the branching after time $T-\varepsilon$. More precisely $X^\varepsilon_t = X_t$ for $t \leq T-\varepsilon$ and $X^\varepsilon_t = X_{T-\varepsilon}P_{T-\varepsilon,t}$ for $t > T-\varepsilon$, where we used the notation $\mu P_{s,t}(dy) = \int \mu(dx)P_{s,t}(x,dy)$. It is hence clear that A_ε meets the assumptions made for (1.9). In fact, in order to see that $u_{A_\varepsilon}(s,t,\,\cdot\,;\,f)$ is uniquely defined by (1.9), observe first that this is clear for $t \leq T-\varepsilon$ or $s \geq T-\varepsilon$. For $t \in (T-\varepsilon, T]$ and $s \in [L, T-\varepsilon)$ note that by (1.9)

$$u_{A_\varepsilon}(s,t,x;\,f) = P_{s,T-\varepsilon}(P_{T-\varepsilon,t}f)(x) - \mathbf{E}_{s,x}\left[\int_s^{T-\varepsilon} A(dr)\psi(r,\xi_r;\,u_{A_\varepsilon}(r,t,\xi_r;\,f))\right].$$

Hence by assumption on A, the solution of this equation is unique and can be expressed as $u_{A_\varepsilon}(s,t,x;\,f) = u_A(s,T-\varepsilon,x;\,P_{T-\varepsilon,t}f)$.)

Lemma 2.1 (Maximum Principle)**.** *Assume that $f_1 \leq f_2$ and $0 \leq \varepsilon_1 \leq \varepsilon_2$. Then*

$$u_{A_{\varepsilon_1}}(s,t,x;\,f_1) \leq u_{A_{\varepsilon_2}}(s,t,x;\,f_2), \qquad s \leq t,\ x \in \mathbb{R}^d. \tag{2.1}$$

Proof. From equation (1.1) it is clear that

$$u_A(s,t,x;\,f_1) \leq u_A(s,t,x;\,f_2). \tag{2.2}$$

This yields the claim for $s \leq t \leq T-\varepsilon_2$. Now let $t \in (T-\varepsilon_2, T]$. For $s \in [T-\varepsilon_2, t]$ clearly

$$u_{A_{\varepsilon_2}}(s,t,x;\,f_2) = P_{s,t}f_2(x) \geq P_{s,t}f_1(x) \geq u_{A_{\varepsilon_1}}(s,t,x;\,f_1). \tag{2.3}$$

For $s \in [L, T-\varepsilon_2)$ use (2.2), (2.3) and the semigroup property to get

$$\begin{aligned} u_{A_{\varepsilon_2}}(s,t,x;\ f_2) &= u_A\big(s, T-\varepsilon_2, x;\ u_{A_{\varepsilon_2}}(T-\varepsilon_2, t,\ \cdot\ ;\ f_2)\big) \\ &\geq u_A\big(s, T-\varepsilon_2, x;\ u_{A_{\varepsilon_1}}(T-\varepsilon_2, t,\ \cdot\ ;\ f_1)\big) \\ &= u_{A_{\varepsilon_1}}(s,t,x;\ f_1). \end{aligned} \tag{2.4}$$

□

Proof of Proposition 1.2. Clearly $\lambda\nu$ is regular whenever $\lambda \in [0,1]$ and ν is regular. Hence for showing existence and uniqueness we can restrict ourselves to the case $\lambda = 1$.

We construct approximate solutions of (1.9) and use the maximum principle to show convergence to a solution as well as uniqueness of the solution. The maximum principle will also be used to compute the derivative at 0 (equation (1.13)).

Existence. Our aim is to construct a solution u_A of (1.9) with f replaced by ν via an approximation scheme. We want to show that for $\varepsilon > 0$ solutions u_{A_ε} of (1.9) with f replaced by ν exist and converge as $\varepsilon \to 0$ to a solution u_A of (1.9).

We can define (recall that $p_{s,T}\nu(x) = \langle \nu, p_{s,T}(x,\ \cdot\ )\rangle$)

$$u_{A_\varepsilon}(s,T,x;\ \nu) = \begin{cases} p_{s,T}\nu(x), & s \geq T-\varepsilon, \\ u_{A_\varepsilon}(s, T-\varepsilon, x;\ p_{T-\varepsilon,T}\nu), & s < T-\varepsilon. \end{cases} \tag{2.5}$$

Clearly $u_{A_\varepsilon}(s,T,x;\ \nu)$ is a solution of (1.9) (A replaced by A_ε and f by ν). Note that by an application of the maximum principle $\varepsilon \mapsto u_{A_\varepsilon}(s,T,x;\ \nu)$ is increasing. Hence we can define $u_A(s,T,x;\ \nu) = \lim_{\varepsilon \downarrow 0} u_{A_\varepsilon}(s,T,x;\ \nu)$ as the pointwise decreasing limit.

Clearly $u_{A_\varepsilon}(s,T,x;\ \nu) \leq p_{s,T}\nu(x)$. Furthermore $\psi(t,x;\ u)$ is a monotone increasing function of u. Thus, by the assumption (1.12) and the dominated convergence theorem we get

$$\begin{aligned} u_A(s,T,x;\ \nu) &= \lim_{\varepsilon \downarrow 0} u_{A_\varepsilon}(s,T,x;\ \nu) \\ &= p_{s,T}\nu(x) - \lim_{\varepsilon \downarrow 0} \mathbf{E}_{s,x}\left[\int_s^T A_\varepsilon(dr)\psi(r,\xi_r;\ u_{A_\varepsilon}(r,T,x;\ \nu))\right] \\ &= p_{s,T}\nu(x) - \mathbf{E}_{s,x}\left[\int_s^T A(dr)\psi(r,\xi_r;\ u_A(r,T,x;\ \nu))\right]. \end{aligned} \tag{2.6}$$

Concluding we see that $u_A(s,T,x;\ \nu)$ is a solution of (1.9).

Uniqueness. Let $v_A(s,x)$, $s \in [L,T]$, $x \in \mathbb{R}^d$, be a solution of (1.9) (with f replaced by ν). Let $\varepsilon > 0$ and note that from (1.9) (and the definition of $u_{A_\varepsilon}(s,t,\ \cdot\ ;\ \nu)$ in (2.5)) it is immediately clear that

$$u_{A_\varepsilon}(T-\varepsilon, T,\ \cdot\ ;\ \nu) \geq v_A(T-\varepsilon,\ \cdot\ ). \tag{2.7}$$

Recall that we did not use uniqueness at the terminal time T to show the semigroup property (1.10) for solutions of (1.9). In particular, for any $r \in (s,T)$ we have

$$v_A(s,x) = u_A(s,r,x;\ v_A(r,\ \cdot\ )), \qquad x \in \mathbb{R}^d. \tag{2.8}$$

Now use the maximum principle applied to the terminal time $T-\varepsilon$ and the functions in (2.7): For all $s < T-\varepsilon$,

$$\begin{aligned} u_{A_\varepsilon}(s,T,\,\cdot\,;\,\nu) &= u_A(s,T-\varepsilon,\,\cdot\,;\,p_{T-\varepsilon,T}\nu) \\ &\geq u_A(s,T-\varepsilon,\,\cdot\,;\,v_A(T-\varepsilon,\,\cdot\,)) \\ &= v_A(s,\,\cdot\,). \end{aligned} \tag{2.9}$$

Thus $u_A \geq v_A$. However, plugging this in the right hand side of (1.9) gives that also $u_A \leq v_A$, thus we have uniqueness.

Derivative at 0. Note that $\psi(r,x;\,0)=0$ and that $u \mapsto \psi(r,x;\,u)$ is convex. Hence $u \mapsto u^{-1}\psi(r,x;\,u)$ is increasing. Further note that

$$\lim_{u\downarrow 0} u^{-1}\psi(r,x;\,u) = 0, \qquad r \in [L,T],\ x \in \mathbb{R}^d. \tag{2.10}$$

Clearly $u_A(r,T,x;\,\lambda\nu) \leq \lambda p_{r,T}\nu(x)$. Hence by assumption (1.12) we can apply the dominated convergence theorem to obtain

$$\begin{aligned} &\limsup_{\lambda\downarrow 0} \lambda^{-1}\mathbf{E}_{s,x}\left[\int_s^T A(dr)\psi(r,\xi_r;\,u_A(r,T,\xi_r;\,\lambda\nu))\right] \\ &\qquad \leq \limsup_{\lambda\downarrow 0} \mathbf{E}_{s,x}\left[\int_s^T A(dr)\lambda^{-1}\psi(r,\xi_r;\,\lambda p_{r,T}\nu(\xi_r))\right] \\ &\qquad = 0. \end{aligned} \tag{2.11}$$

This clearly implies (1.13). □

Proof of Theorem 1. In order to show absolute continuity of X_T it is sufficient to show that $\{y \mapsto \langle X_T, \tau_y\varphi_\gamma\rangle,\ \gamma > 0\}$ is $\mathbf{P}_{L,\mu}$–almost surely uniformly integrable (in $\mathbb{R}^d$). This is clearly the case if it holds for $\mathbf{P}_{L,\delta_x}$ for μ–almost all x. To this end it suffices to show pointwise convergence plus convergence of the mean. Our strategy is to check the assumptions by using the log–Laplace transforms u_A and Proposition 1.2.

Let us first formulate the condition for absolute continuity as a lemma. The simple proof is omitted here (see, for example, [DF95, Lemma 2.7.1]).

Lemma 2.2. *Assume that*

(i) *Z is a random measure on $\mathbb{R}^d$ and $\mathbf{E}[Z]$ is absolutely continuous with density z.*
(ii) *For almost all $y \in \mathbb{R}^d$, the limit in distribution $\tilde\zeta(y) := \lim_{\gamma\downarrow 0}\langle Z, \tau_y\varphi_\gamma\rangle$ exists and $\mathbf{E}[\tilde\zeta(y)] = z(y)$.*

Then for almost all $y \in \mathbb{R}^d$, the almost sure limit $\zeta(y) := \lim_{\gamma\downarrow 0}\langle Z, \tau_y\varphi_\gamma\rangle$ exists and Z is almost surely absolutely continuous with density ζ.

Corollary 2.3. *Assume that (i) holds and that for $\lambda > 0$ and $y \in \mathbb{R}^d$,*

$$v(y,\lambda) = \lim_{\gamma\downarrow 0}\big(-\log(\mathbf{E}[\exp(-\langle Z, \lambda\tau_y\varphi_\gamma\rangle)]\big) \tag{2.12}$$

exists and fulfills

$$\lim_{\lambda\downarrow 0}\lambda^{-1}v(y,\lambda) = z(y) \qquad \text{for almost all } y \in \mathbb{R}^d. \tag{2.13}$$

Then the implication of Lemma 2.2 holds.

Proof. Note that (2.13) implies continuity of $v(y,\lambda)$ at $\lambda = 0$, hence (2.12) implies that ζ exists and that $v(y,\lambda) = -\log \mathbf{E}[\exp(-\lambda\zeta(y))]$. From (2.13) we get $\mathbf{E}[\zeta(y)] = z(y)$. □

In order to proof Theorem 1 we have to check that X_T fulfills the assumptions of the corollary. Apparently $\mathbf{E}_{L,\mu}[X_T] = \mu P_{L,T}$ is absolutely continuous (recall that $\mu P_{L,T}(dy) = \langle \mu, P_{L,T}(\,\cdot\,, dy)\rangle$). To show (2.12) and (2.13) first note that by assumption $\ell(dy)$–almost all δ_y are regular. Hence there exists u_A solving (1.9) with $\nu = \delta_y$. Fix $x, y \in \mathbb{R}^d$ as in Assumption 1.3. With a view to Proposition 1.2, equation (1.13), it is enough to show that

$$u_A(s,T,x;\ \delta_y) = \lim_{\gamma\downarrow 0} u_A(s,T,x;\ \tau_y\varphi_\gamma). \tag{2.14}$$

We proceed similarly as in the proof of Proposition 1.2. Let $\varepsilon > 0$ and note that

$$u_A(s,T,x;\ \tau_y\varphi_\gamma) \le u_{A_\varepsilon}(s,T,x;\ \tau_y\varphi_\gamma), \qquad \gamma > 0, \tag{2.15}$$

and

$$\lim_{\gamma\downarrow 0} u_{A_\varepsilon}(s,T,x;\ \tau_y\varphi_\gamma) = u_{A_\varepsilon}(s,T,x;\ \delta_y). \tag{2.16}$$

Thus

$$\limsup_{\gamma\downarrow 0} u_A(s,T,x;\ \tau_y\varphi_\gamma) \le u_A(s,T,x;\ \delta_y).$$

On the other hand, note that (since $u_{A_\varepsilon} \ge u_A$)

(2.17)

$$\begin{aligned} u_A(s,T,x;\ \tau_y\varphi_\gamma) &\ge P_{s,T}(\tau_y\varphi_\gamma)(x) - \mathbf{E}_{s,x}\left[\int_s^T A(dr)\psi(r,\xi_r;\ u_{A_\varepsilon}(r,T,x;\ \tau_y\varphi_\gamma))\right] \\ &= u_{A_\varepsilon}(s,T,x;\ \tau_y\varphi_\gamma) - \mathbf{E}_{s,x}\left[\int_{T-\varepsilon}^T A(dr)\psi(r,\xi_r;\ P_{r,T}(\tau_y\varphi_\gamma)(\xi_r))\right]. \end{aligned}$$

Thus by (1.14) and (2.16) we have (as in the proof of Proposition 1.2)

$$\begin{aligned} u_A(s,T,x;\ \delta_y) &= \lim_{\varepsilon\downarrow 0} u_{A_\varepsilon}(s,T,x;\ \delta_y) \\ &= \lim_{\varepsilon\downarrow 0}\lim_{\gamma\downarrow 0} u_{A_\varepsilon}(s,T,x;\ \tau_y\varphi_\gamma) \\ &\le \liminf_{\gamma\downarrow 0} u_A(s,T,x;\ \tau_y\varphi_\gamma). \end{aligned} \tag{2.18}$$

This shows (2.14) and finishes the proof. □

Proof of Theorem2. We keep the notation from the previous proofs. Note that with this notation the assumption of the theorem can be written as

$$\lim_{\varepsilon\downarrow 0} u_{A_\varepsilon}(L,T,x;\ \delta_y) = 0, \qquad \mu\otimes\ell\text{–almost all } (x,y). \tag{2.19}$$

Hence by (2.15) and (2.16) for every continuous bounded function $\varphi : \mathbb{R}^d \to [0,\infty)$ with $\int \varphi(x)\,dx = 1$

$$\lim_{\gamma\downarrow 0} u_A(L,T,x;\ \tau_y\varphi_\gamma) = 0, \qquad \mu\otimes\ell\text{–almost all } (x,y). \tag{2.20}$$

As in Corollary 2.3 this implies that $\lim_{\gamma\downarrow 0}\langle X_T, \tau_y\varphi_\gamma\rangle = 0$ in probability. Hence X_T is supported by a Lebesgue null set. □

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